# INTRODUCTION TO HULL-WHITE MODEL

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## 1. Ho-Lee

1.1. **Introduction.** We denote the driving Brownian motion by $W_t$ and work under the risk-neutral measure $\mathbf{Q}$. As we know from [L1], any (one-factor) HJM model is completely determined by the instantaneous forward rate volatility function $\sigma(t,T)$ in the main HJM equation

$$df(t,T) = -\Sigma(t,T)\,\sigma(t,T)\,dt + \sigma(t,T)\,dW_t,$$

where we keep the notation from [L1], in particular

$$\begin{aligned} \sigma(t,T) &= -\frac{\partial}{\partial T}\Sigma(t,T), \\ \Sigma(T,T) &= 0. \end{aligned}$$

In general, $\sigma(t,T)$ can be as complicated as we want it to be. It can depend on $f$, it can be random, etc. But simple models are usually more robust, and anyway we need to understand them before tackling more complex ones.

What is the simplest possible HJM model that we can have? Well, we can set

$$\sigma(t,T) \equiv 0;$$

then

$$df(t,T) = 0$$

and

$$f(t,T) = f(0,T).$$

There is no randomness and every instantaneous forward rate is constant. We move "along" the initial term structure. This is a pretty boring term structure model, albeit occasionally useful in testing.

**HW Problem 1:** Derive the expression for $P(t,T)$ in this trivial model.

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**HW Problem 2:** Show that the model $df(t,T) = m\,dt$ is *not arbitrage-free* for $m \neq 0$ by *explicitly* constructing a zero-cost self-financing portfolio with a non-negative payoff.

Let us rephrase the question. What is the simplest possible **non-trivial** HJM model? We can set

$$\sigma(t,T) \equiv \sigma$$

where $\sigma$ is a given constant – forward rate volatility is constant. This is a valid choice of volatility structure, and we can build a model from it. This model is in fact called the Ho-Lee model.

Historical note: While we derive Ho-Lee from the general HJM specification, historically it happened the other way around. First Ho and Lee formulated their model as a first NO-ARBITRAGE model, and then HJM generalized their idea to a wide class of models.

1.2. **Instantaneous forward rates and a short rate.** It is trivial to solve for $\Sigma$ from the equation

$$\begin{aligned} -\frac{\partial}{\partial T}\Sigma(t,T) &= \sigma, \quad 0 \leq t \leq T, \\ \Sigma(T,T) &= 0. \end{aligned}$$

We have

$$\Sigma(t,T) = -\sigma \times (T-t).$$

Then the equation for $f$'s become

$$df(t,T) = \sigma^2 (T-t)\,dt + \sigma\,dW_t,$$

so that

$$\begin{aligned} f(t,T) &= f(0,T) + \sigma^2 \int_0^t (T-s)\,ds + \sigma \int_0^t dW_s \\ &= f(0,T) + \sigma^2 \left(Tt - t^2/2\right) + \sigma W_t. \end{aligned}$$

Recall our definition of the short rate:

$$r(t) = f(t,t).$$

Substituting $T = t$ in the expression for $f(t,T)$ above we get

$$\begin{aligned} r(t) &= f(t,t) = f(0,t) + \sigma^2 \left(tt - t^2/2\right) + \sigma W_t \\ &= f(0,t) + \frac{\sigma^2 t^2}{2} + \sigma W_t. \end{aligned}$$

It is usually very convenient to separate the initial term structure from the dynamic part. We can do it very easily in our case. Define

(the dynamic part)

$$x(t) = \frac{\sigma^2 t^2}{2} + \sigma W_t$$

and call it a **short-rate state**. (note that the short rate state does not depend on the initial term structure, and $x(0) = 0$. We can think of $x(t)$ as a **deviation** from the initial term structure). Then

$$r(t) = f(0,t) + x(t).$$

We can also express the instantaneous forward rates as a function of the short rate state

$$\begin{aligned} f(t,T) &= f(0,T) + \sigma^2 \left(Tt - t^2/2\right) + \left(x(t) - \frac{\sigma^2 t^2}{2}\right) \\ &= f(0,T) + \sigma^2 t (T - t) + x(t). \end{aligned}$$

The dynamics of the term structure is completely determined by the processes for all its instantaneous forward rates, because everything can be computed using them (spot rates, bonds, forward bonds). Note that all forward rates depend on a single source of noise, $x(t)$. Moreover, $x(t)$ is a Markov process, i.e. if we know the present, the location of $x(t)$, the future is independent of the past. This is a very powerful property that allows us to build **recombining** trees (more on that later).

To emphasize, if we know $x(t)$ at time $t$ we know the whole term structure $\{f(t,T)\}_{T=t}^{\infty}$. We have reduced the evolution of the whole term structure (potentially infinite-dimensional) to the movement of a single stochastic random variable $x(t)$, which in addition follows a very simple process.

1.3. **Bond prices.** Usually it is very useful to have simple expressions for bond prices at future times $P(t,T)$ as functions of some (universal) variables. Seeing that we were successful in expressing all forward rates as functions of a single variable $x(t)$ we can hope to do the same thing for bonds. We have

$$\begin{aligned} P(t,T) &= \exp\left(-\int_t^T f(t,S)\,dS\right) \\ &= \exp\left(-\int_t^T \left[f(0,S) + \sigma^2 t (S-t) + x(t)\right]\,dS\right) \\ &= \exp\left(-\int_t^T f(0,S)\,dS\right) \times \exp\left(\int_t^T \sigma^2 t (S-t)\,dS\right) \\ &\quad \times \exp\left(-(T-t)\,x(t)\right). \end{aligned}$$

The first exponent is just the forward value of the bond at time 0 (known from the initial term structure) $P(0,t,T) = \frac{P(0,T)}{P(0,t)}$. The second exponent is a deterministic drift and can be computed easily; we denote it $A(t,T)$,

$$\begin{aligned} A(t,T) &= \exp\left(\int_t^T \sigma^2 t\,(S-t)\, dS\right) \\ &= \exp\left(\frac{\sigma^2 t}{2}(T-t)^2\right). \end{aligned}$$

And in the second exponent we define

$$b(t,T) = T - t$$

and observe that it is indeed a function of $x(t)$. So we have

$$P(t,T) = P(0,t,T)\, A(t,T)\, e^{-b(t,T)x(t)}$$

and we have succeeded in expressing all bond prices as functions of the same state variable $x(t)$.

## 2. Hull-White

2.1. **Formulas.** The volatility specification from the previous section can be generalized somewhat, with no additional complications, to

$$\sigma(t,T) = \sigma e^{-a(T-t)},$$

where the coefficient $a > 0$ is called the mean reversion coefficient (for reasons explained later). This is the so-called Hull-White model. The formulas for the Hull-White model are as follows.

Volatility functions

$$\begin{aligned} b(t,T) &= \frac{1 - e^{-a(T-t)}}{a}, \\ \Sigma(s,T) &= -\sigma \times b(t,T). \end{aligned}$$

Short rate and instantaneous forward rates

$$\begin{aligned} r(t) &= f(0,t) + x(t), \\ x(t) &= \frac{\sigma^2}{2} b(0,t)^2 + \sigma \int_0^t e^{-a(t-s)}\, W_t, \\ f(t,T) &= f(0,T) + \frac{\sigma^2}{2}\left(b^2(0,T) - b^2(t,T)\right) + \sigma \int_0^t e^{-a(T-s)}\, dW_s, \\ f(t,T) &= f(0,T) + \frac{\sigma^2}{2}\left(b^2(0,T) - b^2(t,T) - e^{-a(T-t)} b^2(0,t)\right) + e^{-a(T-t)} x(t). \end{aligned}$$

Bond prices

$$\begin{aligned} P(t,T) &= P(0,t,T) \times A(t,T)\, e^{-b(t,T)x(t)}, \\ A(t,T) &= \exp\left(-\frac{\sigma^2}{2}\frac{1-e^{-2at}}{2a} b^2(t,T)\right). \end{aligned}$$

**HW Problem 3:** Prove these formulas by repeating the arguments we made for the Ho-Lee model.

2.2. **Distributional properties.** The random variable $x(t)$ is Gaussian. Its variance is given by

$$\operatorname{Var}(x(t)) = \sigma^2 \frac{1-e^{-2at}}{2a}.$$

The bond is log-normal, i.e. the logarithm of the bond is Gaussian. Its variance

$$\operatorname{Var}(\log P(t,T)) = \sigma^2 b^2(t,T)\, \frac{1-e^{-2at}}{2a}.$$

**HW Problem 4:** Prove these formulas.

2.3. **Markovian property.** As we can see all market quantities (forward rates, bond prices) can be expressed in terms of the short rate state $x(t)$, just like in the Ho-Lee model. But is $x(t)$ process Markov? From the formula

$$x(t) = ... + \sigma \int_0^t e^{-a(t-s)}\, W_t$$

it appears that $x(t)$ depends on the whole history of $W$ between 0 and $t$.

Nevertheless, let us compute $dx(t)$. If the drift and diffusion coefficients in the resulting equation depend on quantities at time $t$ only then $x(t)$ is Markov. We have

$$\begin{aligned} dx(t) &= \sigma^2 b(0,t) \frac{d}{dt} b(0,t)\, dt + \sigma d\left(e^{-at} \int_0^t e^{as} dW_s\right) \\ &= \sigma^2 e^{-at} b(0,t)\, dt - a\sigma e^{-at} \left(\int_0^t e^{as} dW_s\right) dt + \sigma e^{-at} e^{at} dW_s \\ &= \sigma^2 e^{-at} b(0,t)\, dt + a\left(\frac{\sigma^2}{2} b^2(0,t) - x(t)\right) dt + \sigma\, dW_s. \end{aligned}$$

After simplifications

$$dx(t) = (\theta(t) - a x(t))\, dt + \sigma\, dW_t, \tag{2.1}$$

where

$$\theta(t) = \sigma^2 e^{-at} b(0,t) + a\frac{\sigma^2}{2} b^2(0,t) = \sigma^2 \frac{1-e^{-2at}}{2a},$$

a deterministic function.

We can see from the equation (2.1) that $x(t)$ is indeed Markov.

2.4. **Mean reversion.** The equation (2.1) makes clear why $a$ is called the mean reversion coefficient. It is proportional to the force with which $x(t)$ is pulled to the "long-term average" $\theta(t)$ (note that $\theta(t)$ is pretty small so we can think of $x(t)$ as reverting to 0. Recall also the interpretation of $x(t)$ as deviation from today's term structure).

Without the stochastic part, the equation

$$dx(t) = -ax(t)\,dt$$

has solutions

$$x(t) = x(0)\,e^{-at},$$

i.e. $x(t)$ experiences exponential "pull to zero" (see Figure 1). When the stochastic term is present, random shocks can move $x(t)$ around but it is still pulled to 0 (or, more accurately, to $\theta(t)$) – see Figure 2.

2.5. **Loadings.** In the Hull-White model

$$\begin{aligned} r(t) &= \ldots + x(t), \\ f(t,T) &= \ldots + e^{-a(T-t)} x(t). \end{aligned}$$

The process $x(t)$ represents a random shock. Recall the statistical model – it can be interpreted as a factor. Then the responses of various rates $f(t,T)$ to this shock are given by the function $e^{-a(T-t)}$. This response can be interpreted, in the framework of the statistical model, as a loading. Note that the longer the time to expiry $T-t$, the smaller the response to the shock.

For spot rates the same qualitative behavior holds:

$$R(t,T) = \frac{1}{T-t}\log P(t,T) = \ldots\frac{b(t,T)}{T-t}x(t),$$

so the loading for the spot rates is given by the function

$$\frac{b(t,T)}{T-t}$$

(compare to the statistical model). For longer $T-t$, the function $\frac{b(t,T)}{T-t}$ is smaller, so again, we see that longer-tenor rates have lower response.

Lower response translates into lower volatility:

$$\operatorname{vol} R(t,T) \propto \frac{b(t,T)}{T-t}.$$

Same holds for forward rates

$$\begin{aligned} F(t,M,T) &= \frac{1}{T-M}\left(\log P(t,T) - \log P(t,M)\right) \\ &= \ldots + \frac{1}{T-M}\left(b(t,T) - b(t,M)\right) x(t) \\ &= \ldots + \frac{1}{T-M}\left(\frac{e^{-a(M-t)} - e^{-a(T-t)}}{a}\right) x(t) \\ &= \ldots + \frac{b(M,T)}{T-M} e^{-a(M-t)} x(t). \end{aligned}$$

The longer the tenor $T-M$, the lower the volatility (everything else being equal).

The results are presented for *instantaneous* volatility but similar results hold for term volatilities.

2.6. **On the importance of mean reversion.** This topic is discussed at length in Rebonato ([RB2]). Models that exhibit mean reversion are generally preferred to those that do not. There are various reasons for that. Some people believe that it is important to recapture this property in the model because the rates in the real world do indeed exhibit mean reversion (see Figure 2). However, this is not a very good argument, because mean reversion in such interpretation is a condition on a drift. However, as we know, the drift gets all changed when we switch to the risk-neutral world, so real-world drift is largely irrelevant to the prices of traded instruments.

However, as pointed by Rebonato, the **drift** of the short rate (non-traded asset) affects the **volatility** of the traded assets, i.e. bonds. It is clear from the formula we derived:

$$\operatorname{vol} R(t,T) \propto \left(\frac{1-e^{-a(T-t)}}{a(T-t)}\right) \tag{2.2}$$

Therefore, having mean reversion allows us to control how volatilities of different bonds relate to each other.

One of the shortcomings of Ho-Lee model is that, given the same short-rate volatility, the volatilities of spot rates are all the same (just plug in $a=0$ in the formula (2.2)). However, longer tenor rates normally have implied volatilities that are lower than shorter-tenor rates, see Figure 3. Examining (2.2) and Figure 4 we see that this can be achieved by cranking the mean reversion up. Thus we need **more** mean reversion to make the volatilities of the rates with longer tenors **lower** than those of shorter tenors.

As a matter of fact "mean reversion" is sometimes understood in exactly this, more broad, sense. The model does not need to have an equation that looks like (2.1) to be "mean reverting". But it does need to have downward sloping loadings. "Mean reversion" than can be understood as the speed of decay in the loading(s).

## 3. Calibration

The usage of any model, and the Hull-White model is no exception, follows the same framework. We calibrate the model to the prices of actively-traded instruments (imply the parameters of the model from the prices of market instruments) and compute prices of the instruments that are not actively traded. By far, the only liquid instruments we can use in calibration are caps and European swaptions. They are usually well represented as a grid (appropriately known as a swaption grid) with swaption's expiry running down and the swap tenor running across. The grid usually contains 100+ swaptions and caps.

In the discussion that follow it is useful to keep in mind the lecture on building smooth term curves. The goal of calibration is the same as the goal of building a term curve – we would like to find internal parameters of the model (forward rates in the case of the curve building) that match externally given market prices.

It is a goal of a calibration of an interest-rate model to match as many swaptions in the grid as possible. The Ho-Lee model has only one parameter, $\sigma$, and therefore can only match a single swaption. Hull-White adds another parameter, $a$, so theoretically we can match two swaptions (However, a more useful way to "spend" the parameter $a$ is to get a general shape of a loading right.)

Many instruments, with a notable presence of Bermuda swaptions, require simultaneous calibration to many European swaptions. The straight Hull-White model does not allow us that flexibility. However, the Hull-White can be extended even further to introduce time-dependent volatility $\sigma(t)$ and even time-dependent reversion $a(t)$.

Formally, the extended Hull-White model is characterized by the volatility structures of forward rates of the form

$$\sigma(t,T) = \sigma(t)\, e^{-\int_t^T a(u)\,du}.$$

Most of the formulas we have derived can be generalized in the extended Hull-White model. While we lose nice closed-form solutions, and (more importantly) time-homogeneity, we keep important features of the model – it is still a one-dimensional Gaussian Markov model.

In this model we treat $\sigma(t)$ and $a(t)$ as unknown parameters to be found during calibration. As previously discussed, adding more parameters to any model allows for more flexibility so that we can fit more instruments, but may result in an overfitted model if we are not careful. The symptoms of an overfitted model are parameters that defy intuition, are time-inhomogeneous and unstable in day-to-day recalibrations. **It is very easy to overfit the extended Hull-White model**.

While we can stabilize the calibration of the extended Hull-White model using numerical tricks (including smoothing objectives and the like), we should realize that the goal of fitting the whole swaption grid (or a significant portion of it) is too ambitious for any one-factor model, Hull-White included. One-factors-ness is just too-strong a handicap to overcome. We shall examine this topic in detail later in the class.

## References

[L1] Vladimir Piterbarg. Heath-Jarrow-Morton framework. Lecture notes, 2000.
[SC] Lecture notes on stochastic calculus, 1999.
[RB2] Riccardo Rebonato. Volatility and Correlation. Wiley 1999.

# 4. Appendix

**Figure 1**

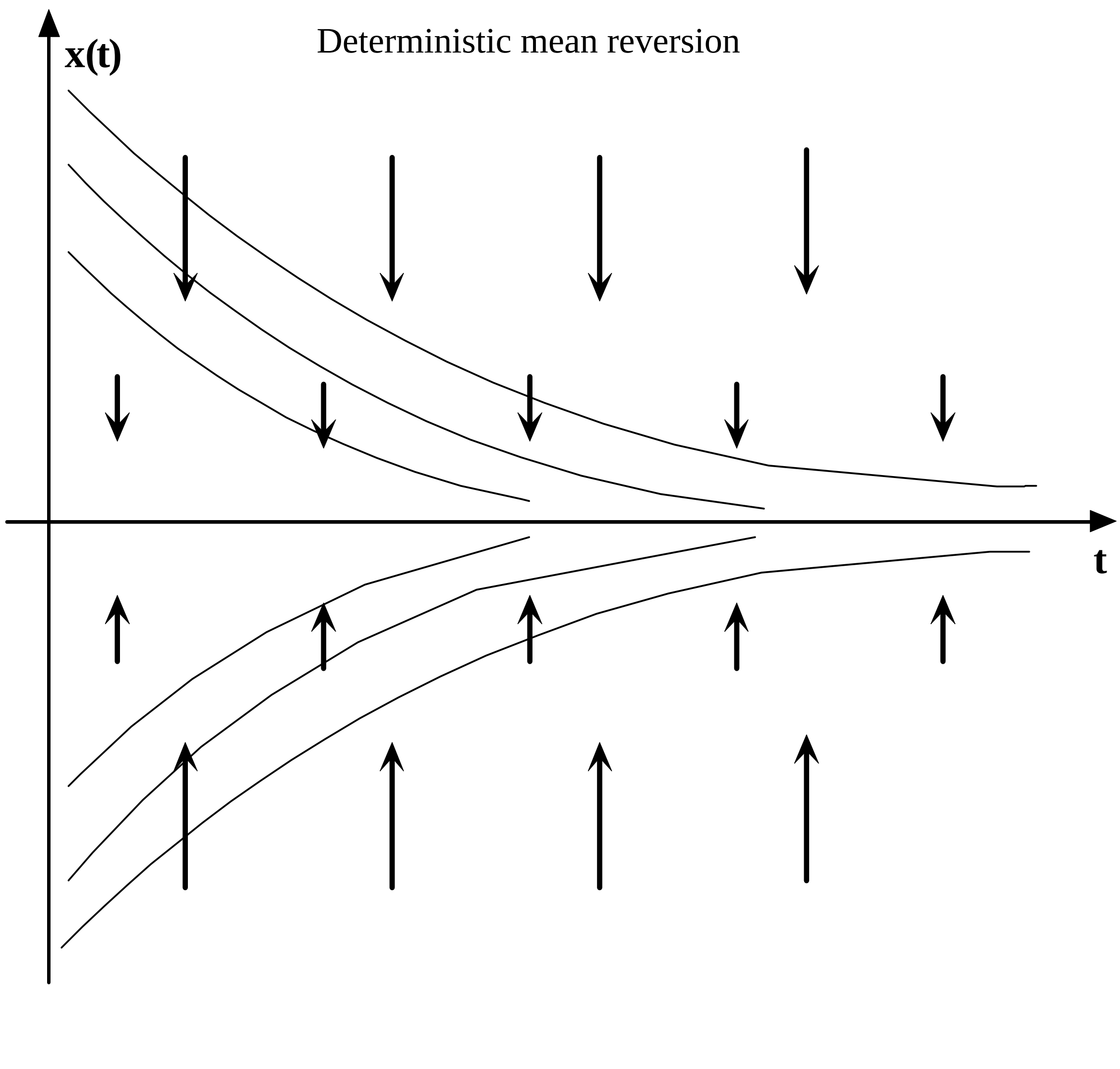

**Figure 2**

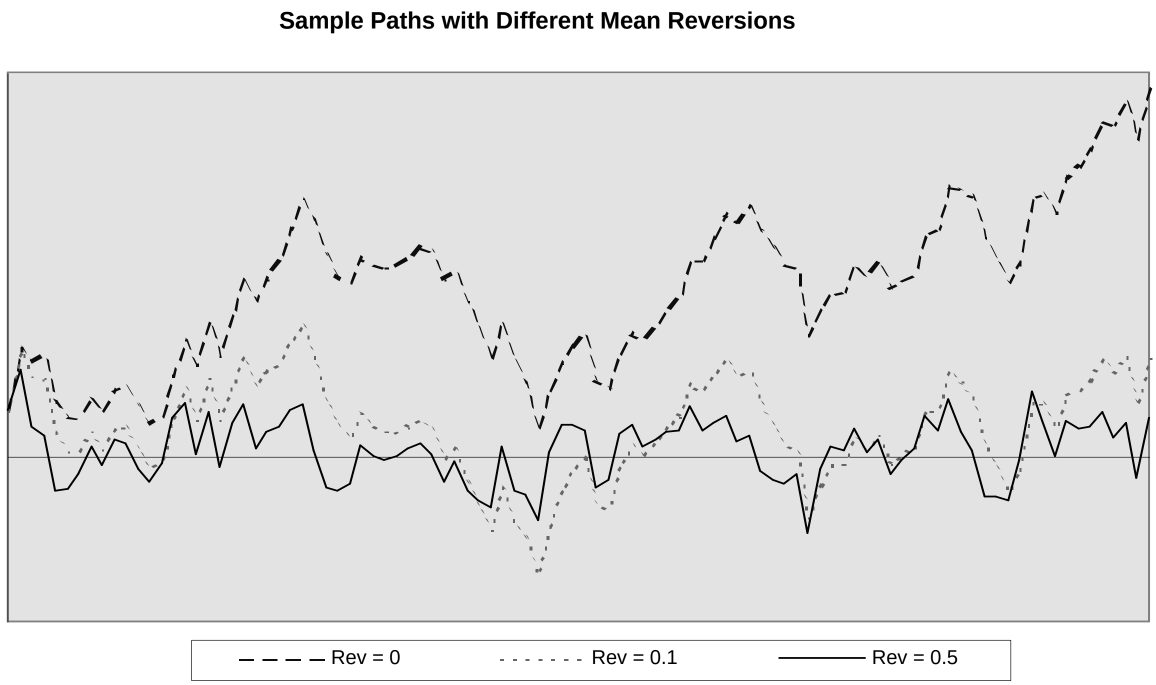
Sample Paths with Different Mean Reversions
Rev = 0
Rev = 0.1
Rev = 0.5

**Figure 3**

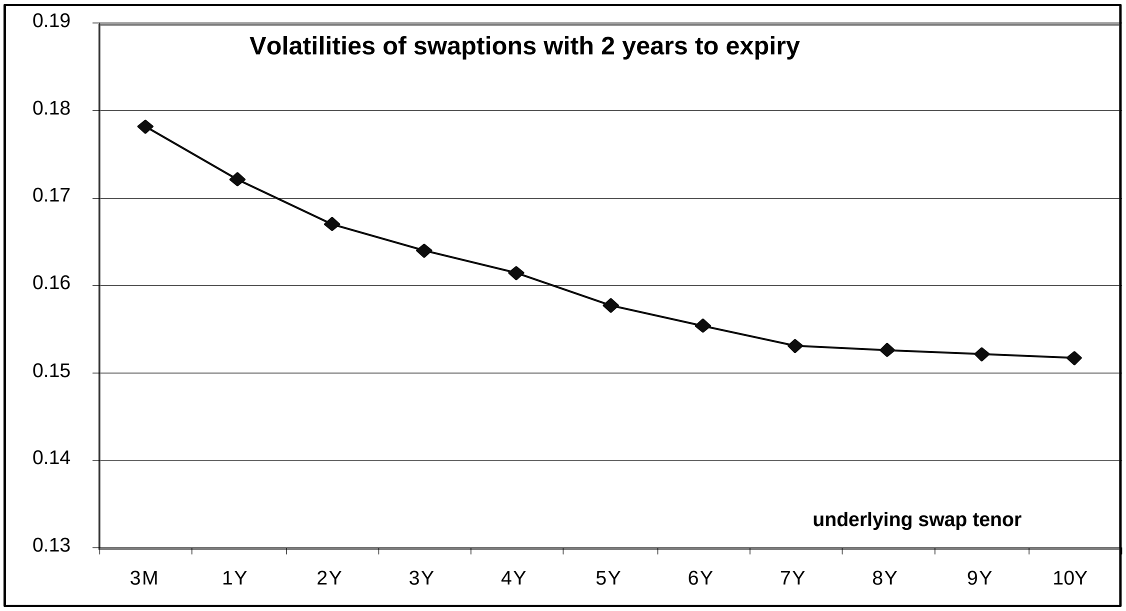


**Figure 4**