LIBOR MARKET MODELS

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1. Brief history lesson

In the beginning of time, interest-rate options (caps/floors and swaptions) were valued using Black's model. Black's model was fast. Black's model was well-understood by all, including traders. And Black's model quickly became "industry standard" for these instruments.

Then academicians came. They felt uneasy about models that had not been derived from the "first principles". The no-arbitrage paradigm was applied to interest rate markets. It all culminated with the creation of HJM framework.

To academicians' dismay, traders kept using their beloved Black's model for valuing caps and swaptions. The reasons were pretty clear. The "stochastic drivers" of Black's model (LIBOR and swap rates) were easily observable, and so were their volatilities. On the contrary, the stochastic drivers of HJM models (instantaneous forward rates) were not directly observable, and neither were their volatilities. A Quant equipped with an HJM model was forced to constantly perform translations between observable quantities and his model's input parameters (a process known as calibration). For most of the models, calibration had proved to be too much of a chore, and easily exceeded an average trader's patience and knowledge base.

And then academicians, who by that time moved to Wall Street, had a bright idea. They embarked on a quest to create an HJM, no arbitrage model in which caps/swaptions would be valued using Black-like expressions, and where the input parameters would be directly observable on the market.

And so the breed of "market models" was born.

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2. Introduction

Black's model for caps is derived under the assumption that LIBOR rates (corresponding to all caplets in the cap) are lognormally distributed. Likewise, Black's model for swaptions is derived under the assumption that the corresponding swap rate is log-normally distributed.

Any no-arbitrage model in which some of the LIBOR or swap rates are log-normally distributed deserves the name of a "market model".

More broadly, any (no-arbitrage) model where the input parameters are given in terms of "observables" (a subset of LIBOR and swap rates) can also be called a market model.

For the purposes of this lecture we will understand "no arbitrage model" as "HJM model". The class of no-arbitrage models is quite a bit broader than the class of HJM models, but HJM models provide very convenient technical tools (Ito calculus in particular) to be ignored.

Any HJM model is uniquely defined by the volatility structure of instantaneous forward rates (see [L1]). We will build market models by choosing this volatility structure in such a way that observable rates (LIBOR and/or swap) have the dynamics we want. This is our general plan of attack.

Let us recall the notations. Risk-neutral measure is denoted by $\mathbf{Q}$. Brownian motion (under $\mathbf{Q}$) is denoted by W_t (once again we consider one-factor models to save some trees). The model is specified by

$$\begin{aligned} df(t,T) &= -\Sigma(t,T)\,\sigma(t,T)\,dt + \sigma(t,T)\,dW_t, \\ dP(t,T) &= r(t)\,P(t,T)\,dt + P(t,T)\,\Sigma(t,T)\,dW_t, \end{aligned}$$

where

$$\frac{\partial \Sigma(t,T)}{\partial T} = -\sigma(t,T).$$

The volatility structure $\sigma(\cdot,\cdot)$ is as yet unspecified.

3. Simple example

Let us consider a single LIBOR rate, and try to choose volatility structure in such a way that the option on this LIBOR rate (caplet) is priced using Black's formula.

The fixing date of the caplet is set to T, and the tenor to δ. We define the forward LIBOR rate $L(t)$ by

$$L(t) = \frac{P(t,T) - P(t,T+\delta)}{\delta P(t,T+\delta)}.$$

A caplet (call option on L) pays

$$(L(T) - K)^+$$

at time $T+\delta$ (note that the payoff $(L(T)-K)^+$ is determined at time T, so that it is $\mathcal{F}_T$-measurable, yet it is paid later at $T+\delta$). Then ($c(t)$ is the caplet's value at time t)

$$\begin{aligned} c(t) &= B_t \mathbf{E}\left(B_{T+\delta}^{-1} c(T) \middle| \mathcal{F}_t \right) \\ &= B_t \mathbf{E}\left(B_{T+\delta}^{-1} (L(T) - K)^+ \middle| \mathcal{F}_t \right). \end{aligned}$$

Let us change the measure to $T+\delta$-forward. We have,

$$c(t) = P(t, T+\delta)\, \mathbf{E}^{T+\delta} \left((L(T) - K)^+ \middle| \mathcal{F}_t \right).$$

Note that $L(t)$ is the value of a traded asset $(\delta^{-1}(P(t,T) - P(t,T+\delta)))$ divided by the numeraire $P(t,T+\delta)$; hence $L(t)$ is a martingale under $\mathbf{Q}^{T+\delta}$.

Black formula can be derived if we assume that ($\hat{W}$ is a Brownian motion in some measure)

$$\begin{aligned} L(t) &= L(0) \exp\left(\lambda \hat{W}_t - \lambda^2 t/2 \right), \\ dL(t) &= \lambda L(t)\, d\hat{W}_t. \end{aligned} \tag{3.1}$$

Note that the process defined by (3.1) is a martingale. So at least our finding is consistent with our goal.

What is the equation for $L(t)$ under $\mathbf{Q}^{T+\delta}$ in our HJM model? For a forward bond $F(t,S,M) = P(t,M)/P(t,S)$ we have (see homework #VP1 or ([L2]))

$$\begin{aligned} dF(t,S,M) &= F(t,S,M)\, \gamma(t,S,M)\, dW_t^S, \\ \gamma(t,S,M) &= \Sigma(t,M) - \Sigma(t,S). \end{aligned}$$

For a LIBOR rate we have (formally $F(t,T+\delta,T)$ is not defined since the second argument is larger than the third, but all formulas are still valid)

$$L(t) = \delta^{-1}(F(t,T+\delta,T) - 1)$$

and

$$dF(t,T+\delta,T) = F(t,T+\delta,T)\, \gamma(t,T+\delta,T)\, dW_t^{T+\delta},$$

where

$$\gamma(t,T+\delta,T) = \Sigma(t,T) - \Sigma(t,T+\delta),$$

so that

$$\begin{aligned} dL(t) &= \delta^{-1}\, dF(t, T+\delta, T) \\ &= \delta^{-1} F(t, T+\delta, T)\left[\Sigma(t,T) - \Sigma(t, T+\delta)\right]\, dW_t^{T+\delta} \\ &= \left(L(t) + \delta^{-1}\right)\left[\Sigma(t,T) - \Sigma(t, T+\delta)\right]\, dW_t^{T+\delta}. \end{aligned}$$

Then under $\mathbf{Q}^{T+\delta}$ measure,

$$dL(t) = \frac{\delta L(t) + 1}{\delta L(t)}\left[\Sigma(t,T) - \Sigma(t, T+\delta)\right] L(t)\; dW_t^{T+\delta}. \tag{3.2}$$

Let us compare what we want (equation (3.1)) and what we have ((3.2)). We can identify $W_t^{T+\delta}$ and $\hat{W}_t$. Then, as long as we choose $\Sigma(t,T)$ and $\Sigma(t,T+\delta)$ such that

$$\frac{\delta L(t) + 1}{\delta L(t)}\gamma(t, T+\delta, T) = \lambda, \tag{3.3}$$

we are assured that

$$dL(t) = \lambda L(t)\; dW_t^{T+\delta},$$

and the caplet on $L(t)$ is priced using Black's formula (with volatility λ):

$$\begin{aligned} c(0) &= P(0, T+\delta)\, \mathbf{E}^{T+\delta}\left((L(T) - K)^+\right) \\ &= P(0, T+\delta)\, \mathbf{E}^{T+\delta}\left(\left(L(0)\, e^{\lambda W_T^{T+\delta} - \lambda^2 T/2} - K\right)^+\right). \end{aligned}$$

So far, we have been presenting the motivation for why market models can be constructed. Now let us present the actual construction of the model in which $L(t)$ has a lognormal distribution. We will pretty much retrace the steps we have outlined above.

1. Specify (observe on the market) a caplet's volatility λ.
2. Specify the dynamics of the LIBOR rate $L(t)$ under $T+\delta$-forward measure $\mathbf{Q}^{T+\delta}$ by
$$dL(t) = \lambda L(t)\; dW_t^{T+\delta},$$
so that
$$L(t) = L(0)\, e^{\lambda W_t^{T+\delta} - \lambda^2 T/2}.$$
3. Define the volatility of the forward bond by
$$\gamma(t, T+\delta, T) = \lambda \times \frac{\delta L(t)}{\delta L(t) + 1}.$$
Note that $\gamma(\cdot, T, T+\delta)$ is an adapted process.

4. Define
$$\gamma(t, T+\delta, T) = \Sigma(t, T) - \Sigma(t, T+\delta)$$
$$= \int_T^{T+\delta} \sigma(t, u)\, du.$$
Choose $\sigma(t, u)$ constant for $u \in [T, T+\delta]$ so that the equation
$$\lambda \times \frac{\delta L(t)}{\delta L(t) + 1} = \int_T^{T+\delta} \sigma(t, u)\, du$$
is satisfied; namely, take
$$\sigma(t, u) = \frac{1}{\delta} \lambda \frac{\delta L(t)}{\delta L(t) + 1}$$
$$= \frac{\lambda L(t)}{\delta L(t) + 1}.$$
5. For $u \notin [T, T+\delta]$ choose $\sigma(t, u)$ arbitrarily. For example, set
$$\sigma(t, u) = \frac{\lambda L(t)}{\delta L(t) + 1}$$
for **all** u, or even
$$\sigma(t, u) \equiv 0$$
for $u \notin [T, T+\delta]$ (which of he choices is more "realistic"?)
6. Now the model is completely specified under $T+\delta$-forward measure $\mathbf{Q}^{T+\delta}$. Change it to risk-neutral and that is it.

4. Market model of LIBOR rates (BGM/J)

In the previous section we constructed an HJM model where a single LIBOR rate followed a lognormal process. It is possible to extend that on a collection of LIBOR rates. Such a model is called BGM/J (Brace-Gatarek-Musiela/Jamshidian).

Fix a tenor structure
$$\begin{aligned} T_0 &= 0 < T_1 < \cdots < T_M, \\ \delta_m &= T_{m+1} - T_m. \end{aligned}$$
Consider a collection of LIBOR rates
$$\{L_m(t)\}_{m=1}^{M-1},$$
where for each m, $L_m(t)$ is a LIBOR rate that resets at T_m and with tenor δ_m (so the corresponding floating cashflows pays at T_{m+1}), so

that

$$L_m(t) = \frac{P(t,T_m) - P(t,T_m+\delta)}{\delta_m P(t,T_m+\delta)} = \frac{P(t,T_m) - P(t,T_{m+1})}{\delta_m P(t,T_{m+1})}.$$

Suppose a collection of caplet volatilities $\{\lambda_m\}_{m=1}^{M-1}$ that we want to match is fixed (observed on the market). Then the following theorem holds.

Theorem 4.1 (LIBOR market model)**.** *There exists an HJM model on a probability space $(\ ,\mathcal{F})$ with risk-neutral measure $\mathbf{Q}$ such that for every m, $m = 1,\ldots,M-1$,*

$$dL_m(t) = \lambda_m L_m(t)\, dW_t^{T_{m+1}},$$

where $W_t^{T_{m+1}}$ is a Brownian motion under T_{m+1}-forward measure $\mathbf{Q}^{T_{m+1}}$.

Proof. Goes pretty much like the one in our simple example for a single LIBOR rate. For details see [MR, Chapter 14]. ∎

Each LIBOR rate follows a lognormal process *under its own measure* $\mathbf{Q}^{T+\delta}$. This guarantees that Black's assumptions are satisfied. However, it makes evaluation of instruments that depend on *more than one* LIBOR rate quite difficult. It would be much more convenient if we knew the simultaneous dynamics of *all* LIBOR rates under a *single* measure. It turns out that risk-neutral measure is not a convenient measure for LIBOR market models, so something else would be useful.

Jamshidian (see [J]) was probably the first one to construct such a universal measure. He called it a *spot LIBOR measure.*

Recall that risk-neutral measure corresponds to the choice of money-market account as a numeraire. In a money market account, the money is constantly reinvested at short rate.

Spot LIBOR measure corresponds to a "discretely compounded numeraire". The money is reinvested at LIBOR rates at times T_m for the next time period $[T_m, T_{m+1}]$. if we start with \$1 at time T_0, then the value of the discretely-compounded money-market account is given by

$$\begin{aligned} G_{T_0} &= 1, \\ G_{T_1} &= G_{T_0}\left(1+\delta_0 L_0(T_0)\right), \\ G_{T_2} &= G_{T_1}\left(1+\delta_1 L_1(T_1)\right), \\ &\ldots. \end{aligned}$$

Note that

$$1+\delta_j L_j(T_j) = \frac{1}{P(T_j,T_{j+1})}$$

so that

$$G_{T_m} = \prod_{j=0}^{m-1} \left(1 + \delta_j L_j \left(T_j\right)\right) = \prod_{j=1}^{m} P^{-1} \left(T_{j-1}, T_j\right).$$

In between "rollover" dates $\{T_m\}_{m=0}^{M}$, G_t is uniquely specified by no-arbitrage arguments. If

$$T_{m-1} < t < T_m,$$

then

$$G_t = P\left(t, T_m\right) \cdot G_{T_m}.$$

Define a deterministic function ("index of a first rollover date after t")

$$m\left(t\right) = \inf\left\{k \in \mathbb{Z} : T_k \geq t\right\}.$$

Then

$$G_t = P\left(t, T_{m(t)}\right) \prod_{j=1}^{m(t)} P^{-1}\left(T_{j-1}, T_j\right).$$

Definition 4.1. *A* spot LIBOR measure $\bar{\mathbf{Q}}^L$ *is a measure that corresponds to G_t being a numeraire, namely the measure under which*

$$\frac{P\left(t, T_m\right)}{G_t}$$

is a martingale for each T_m, $m = 1, \ldots, M$.

Theorem 4.2 (On spot LIBOR measure)**.** *The dynamics of the LIBOR rates under spot LIBOR measure $\bar{\mathbf{Q}}^L$ are given by*

(4.1)

$$dL_j\left(t\right) = \sum_{k=m(t)}^{j} \frac{\delta_{k+1}\lambda_k\lambda_j L_k\left(t\right) L_j\left(t\right)}{1 + \delta_{k+1} L\left(t, T_k\right)} dt + \lambda_j L_j\left(t\right) d\bar{W}_t^L, \quad j = 1, \ldots, M-1,$$

where $\bar{W}_t^L$ is a Brownian motion under $\bar{\mathbf{Q}}^L$.

LIBOR rates are of course no longer martingales under $\bar{\mathbf{Q}}^L$. However, the drifts in (4.1) are still expressed in terms of "observables". An important practical conclusion is that once we specify caplet volatilities $\{\lambda_m\}_{m=1}^{M-1}$, we do not have to backup instantaneous forward volatilities $\sigma\left(\cdot, \cdot\right)$ from them; we can use $\{\lambda_m\}_{m=1}^{M-1}$ and (4.1) to completely and consistently specify the evolution of LIBOR rates $\{L_m\left(t\right)\}_{m=1}^{M-1}$ under the same measure.

Also note that the numeraire G_t is also specified in terms of observables, so we can "evolve forward" LIBOR rates and the numeraire simultaneously (say in Monte-Carlo), only knowing caplet volatilities.

5. Model or parametrization?

In market models, volatilities of LIBOR rates need not be constant, nor the number of factors need be 1. In the most general form we can write the following joint system for stochastic evolution of LIBOR rates $\{L_j(t)\}_{j=1}^{M-1}$,

(5.1)

$$dL_j(t)/L_j(t) = \mu_j(t)\ dt + \sum_{n=1}^{N} a_{jn}(t)\ dZ_n(t), \quad j = 1, \ldots, M-1.$$

It does not really matter what measure we use; this general form will hold under any measure. The difference between different measures will be in different drifts $\{\mu_j(t)\}_{j=1}^{M-1}$, that are in general will not be deterministic functions, but some rather complex expressions involving rates, volatilities, correlations, etc.

In the formula (5.1), N is the number of factors, $\{Z_n\}_{n=1}^{N}$ are independent Brownian motions (under whatever measure we are working in), $a_{jn}(t)$ are deterministic, time-dependent, instantaneous factor volatilities. LIBOR rate instantaneous volatility $v_j(t)$ for the j-th rate $L_j(\cdot)$ is defined by

$$v_j^2(t) = a_{j1}^2(t) + \cdots + a_{jN}^2(t).$$

> **Calibration** is a process of specifying the time-dependent matrix of instantaneous volatilities $\{a_{jn}(t); \quad j = 1, \ldots, M-1, \quad n = 1, \ldots, N, \quad t \geq ($ so that market prices of some instruments are recovered by the model.

The number of degrees of freedom in the model specified by (5.1) is enormous. Let us compute the number of free parameters. Without loss of generality, the instantaneous volatilities $a_{jn}(\cdot)$ can be taken to be constant on time intervals $[T_m, T_{m+1}]$. So for each $a_{jn}(\cdot)$ we have as many degrees of freedom as there are intervals before the fixing of the j-th rate. For the LIBOR rate $L_j(t)$, there are j periods between today (T_0) and the LIBOR fixing date T_j. So there are $j \times N$ degrees of freedom (N is the number of factors) associated with the rate $L_j(\cdot)$. For the total of $M-1$ LIBOR rates there are

$$N \sum_{j=1}^{M-1} j = \frac{1}{2} NM(M-1)$$

degrees of freedom total. Assuming 30 years span that the model has to cover and the LIBOR tenor of 3 months we have $M = 120$. Given say three factors we have about 21420 degrees of freedom (!!).

The number of traded instruments (traded actively enough to matter) is much smaller – there are maybe at most 10 caps and 20 swaptions that are very liquid and maybe 20 more swaptions that are somewhat liquid. It is clear that in a calibration, a huge number of model parameters will be left "unfixed", given a relatively paltry universe of traded instruments. This is another classic case of a potential OVERFITTING!

At this point we should realize that the expression (5.1) is **not** really a model. At least not a model in the sense we come to understand it. In particular, as presented,

- the "model" (5.1) does not provide any structure to the dynamics of rates;
- the "model" (5.1) has no predictive power;
- the "model" (5.1) has no potential to show any mispricing in the market because it will happily match whatever prices are thrown at it.

More on this subject can (and should!) be read in [R1, Chapter 18.5].

We should regard the expression (5.1) as merely a new and (arguably) better **parametrization** of the dynamics of the interest rates. The original parametrization of HJM was done in terms of instantaneous forward rates – the equation (5.1) is just a reparametrization in terms of other (namely LIBOR) rates. *The name "market model" is quite misleading – what we have is "market parametrization".* The new parametrization is considered better because it allows traders to express their opinion using familiar notions of caplet volatilities, correlations between LIBOR rates, etc., and not in the much-less intuitive concepts related to instantaneous forward rates. Still, an opinion MUST be expressed one way or another, as the model itself does not impose any.

A parametrization becomes a model when we add assumptions (or opinions; or views; these are all synonyms) to it (again, read [R1, Chapter 18.5]). The parametrization has to be coupled with something else, something that provides a sensible and stable description of the term structure of caplet volatilities/correlations. This something has to come from an *external* source. The assumptions reduce the dimensionality of the space of model parameters (from 21420 to a more manageable number) and as such allow us

- To express a view on the dynamics of rates;

- To compare the actual evolution with predicted and identify mispricings and trading opportunities;
- To have a stable set of input parameters that does not change unpredictably day-to-day; when it does change it sends a powerful and important signal (change of regime, etc.).

Recall the statistical model of Yuri. In it, a potentially huge number of input parameters was reduced to three by a clever application of statistical analysis and fixing the majority of the parameters from considerations other than the prices of market instruments.

Note that a similar approach can be adapted to the market models. We can set

$$a_{jn}(t) = \phi_n \mathfrak{L}_n (T_j - t)$$

so that

$$dL_j(t)/L_j(t) = \mu_j(t)\ dt + \sum_{n=1}^{N} \mathfrak{L}_n (T_j - t)\,\phi_n\, dZ_n(t), \quad j = 1, \dots, M-1$$

and we have the same interpretation – $\mathfrak{L}_n(\tau)$ is the n-th loading of LIBOR rates. The loadings can be estimated statistically (from the history of LIBOR rates instead of zero rates as in Yuri's model) and ϕ's can be implied from the market.

We have barely scratched the surface of a very complex issue of calibrating market models. Reading Rebonato's books [R1] and [R2] is mandatory for anybody interested in the subject. Also, Appendix gives an illuminating example.

6. Calibration

6.1. **Philosophy.** While the complete flexibility of market models is a mixed blessing (see previous section) it does give us hope to match the prices of essentially all caps and swaptions simultaneously. This is a very serious promise as few, if any, models had attempted to do that before the market models were introduced. This is also the reason why the market models were embraced so enthusiastically by the practitioners. It gives everybody a certain piece of mind knowing that the prices of all the actively traded instruments is matched by their model. (this can be quite misleading however, as you hopefully realize by now.)

While we strongly argue for imposing rigid parametric constraints on market models (5.1), this should not impede the model's ability to match all caps/swaptions. The balance between low dimensionality of parameters and matching all traded instruments is very hard to obtain, however. We will not try to address this issue in the lecture. We will concentrate on deriving formulas needed for calibration.

Normally, a market model would be calibrated to caps and swaptions. Caplets can be valued in the market model using Black's formula. Swaptions are a bit trickier (we will discuss them later). In addition, the model can be calibrated to traders's specific views on

- Correlation of LIBOR rates;
- Correlation of swap rates;
- Forward volatility of LIBOR and swap rates (volatility of rates over a time horizon in the future).

All these quantities can be expressed in terms of input (instantaneous caplet) volatilities relatively easily. Below we show how.

6.2. **Caplets.** We generally want to make sure that the prices of caplets that correspond to the rates $\{L_j(t)\}_{j=1}^{M-1}$ are recovered. This is relatively easy. To match the price of a caplet we just need to make sure that the total volatility of a specific LIBOR rate, as given by the model, matches the caplet Black's volatility. The following constraints have to be satisfied:

$$\mathrm{Var}\left[\log L_j(T_j)\right] = \lambda_j^2 T_j, \quad j = 1, \ldots, M-1.$$

From (5.1), the variances are easily calculated,

$$\int_0^{T_j} v_j^2(t)\, dt = \int_0^{T_j} \left[\sum_{n=1}^{N} a_{jn}^2(t)\right] dt = \lambda_j^2 T_j, \quad j = 1, \ldots, M-1.$$

6.3. **Swaptions.** Black's model for swaptions is considered industry-standard to the point that in the inter-dealer market they are quoted in terms of Black's volatilities. The Black's model is of course based on the assumption that the swap rate is log-normally distributed. It can be shown however that in the model (5.1) swap rates are NOT lognormal. However, at least for relatively short-dated swaptions, the swap rates can be considered approximately lognormal. Then the question becomes, how to identify the Black's volatility for a swap rate that is consistent with our market model.

Let the swap begin at period b and end at period e (so that the first fixing is on T_b and the last payment is on T_{e+1}). Then the swap rate is defined by

$$S_{b,e}(t) = \frac{P(t, T_b) - P(t, T_{e+1})}{\sum_{k=b}^{e} \delta_k P(t, T_{k+1})}.$$

The following simple-minded but surprisingly effective approximation is normally used. We have

$$
\begin{aligned}
S_{b,e}(t) &= \frac{P(t,T_b)-P(t,T_{e+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})} \\
&= \frac{P(t,T_b)-P(t,T_{b+1})+P(t,T_{b+1})-\cdots+P(t,T_e)-P(t,T_{e+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})} \\
&= \frac{P(t,T_b)-P(t,T_{b+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})}+\frac{P(t,T_{b+1})-P(t,T_{b+2})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})}+\dots \\
&\quad +\frac{P(t,T_e)-P(t,T_{e+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})} \\
&= \frac{\delta_b P(t,T_{b+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})}\times\frac{P(t,T_b)-P(t,T_{b+1})}{\delta_b P(t,T_{b+1})}+\dots \\
&\quad +\frac{\delta_e P(t,T_{e+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})}\times\frac{P(t,T_e)-P(t,T_{e+1})}{\delta_e P(t,T_{e+1})}.
\end{aligned}
$$

We define

$$
w_l(t)=\frac{\delta_{l+1}P(t,T_{l+1})}{\sum_{k=b}^{e}\delta_k P(t,T_{k+1})}.
$$

Note that

$$
\begin{aligned}
w_l(t) &> 0, \\
\sum_{l=b}^{e} w_l(t) &= 1.
\end{aligned}
$$

Therefore

$$
S_{b,e}(t)=w_b(t)\times L_b(t)+\cdots+w_e(t)\times L_e(t).
$$

Hence, a swap rate is a weighted sum of LIBOR rates. Note that the weights are time dependent and up to this point we have made no approximations. Now we make one – we say

$$
w_l(t)\approx w_l(0)
$$

for all l. Note that $w_l(0)$ can be determined from today's term curve. The rationale for this approximation is that the weights w are much less volatile than the rates L involved in the formula.

Finally we have an approximation formula (we drop (0) from the weights)

$$
S_{b,e}(t)\approx w_b\times L_b(t)+\cdots+w_e\times L_e(t). \tag{6.1}
$$

Before computing the volatility, recall the normal-lognormal approximation for volatility and variance that Brian presented a while ago. For a lognormal process X_T we have

$$\mathrm{Var} X_T \approx (X_0)^2 \, \mathrm{Var} \log X_T. \tag{6.2}$$

The volatility of $S_{b,e}(T_b)$ (of its log actually as we use lognormal assumptions) can now be computed. We will use the approximation (6.2) repeatedly, first for the swap rate and then for Libor rates.

(6.3)

$$\begin{aligned}
\mathrm{Var} \log S_{b,e}(T_b) &\approx \frac{1}{S_{b,e}^2(0)} \mathrm{Var} S_{b,e}(T_b) = \frac{1}{S_{b,e}^2(0)} \mathrm{Var}\left(\sum_{k=b+1}^{e} w_k \times L_k(T_b) \right) \\
&= \frac{1}{S_{b,e}^2(0)} \sum_{k,l=b+1}^{e} w_k w_l \mathrm{covar}\left(L_k(T_b), L_l(T_b)\right) \\
&= \sum_{k,l=b}^{e} w_k w_l \frac{L_k(0) L_l(0)}{S_{b,e}^2(0)} \mathrm{covar}\left(\log L_k(T_b), \log L_l(T_b)\right) \\
&= \sum_{k,l=b}^{e} w_k w_l \frac{L_k(0) L_l(0)}{S_{b,e}^2(0)} \int_0^{T_b} \left[\sum_{n=1}^{N} a_{kn}(t) a_{ln}(t) \right] dt.
\end{aligned}$$

This formula, albeit formidable in appearance, is quite easy to code up. To value swaptions in the market model we use Black's formula with the volatility given by this expression.

6.4. **Correlation of LIBOR rates.** Sometimes a trader may have a view on what the correlation of LIBOR rates is – either from "gut feeling" or from (infrequently) observed prices of spread options. The correlation can be added to the list of calibration targets. Instantaneous covariance of LIBOR rates j and m is given by

$$\mathrm{covar}\left[\frac{dL_j(t)}{L_j(t)}, \frac{dL_m(t)}{L_m(t)} \right] = \sum_{n=1}^{N} a_{jn}(t) a_{mn}(t).$$

A term covariance or correlation is more likely to be a number we can have an opinion about. The covariance is obtained by integrating the previous formula over a given time interval:

$$\mathrm{covar}\left[\log L_j(T), \log L_m(T)\right] = \sum_{n=1}^{N} \int_0^{T} a_{jn}(t) a_{mn}(t) \, dt.$$

A correlation is obtained by dividing by the appropriate (term) volatilities

$$\operatorname{Corr}\left[\log L_j\left(T\right), \log L_m\left(T\right)\right] = \frac{\sum_{n=1}^{N} \int_0^T a_{jn}\left(t\right) a_{mn}\left(t\right)\, dt}{\left(\int_0^T v_j^2\left(t\right)\, dt\right)^{1/2} \left(\int_0^T v_m^2\left(t\right)\, dt\right)^{1/2}}.$$

6.5. **Correlation of swap rates.** Similar to the way a volatility of swap rates was derived in (6.3), a correlation of swap rates can be obtained from the assumption (6.1). We do not present the formula here because of its cumbersomeness (see ... for details if interested).

6.6. **Forward volatility of LIBOR rates.** A very important point to remember is that the value of the caplet on rate L_j depends on **integrated** (or **term**) volatility of L_j between 0 and T_j:

$$\text{caplet Black's vol } = \left(\frac{1}{T_j} \int_0^{T_j} v_j^2\left(t\right)\, dt\right)^{1/2}.$$

In particular the market (in caps at least) has no information on the volatility of the rate over a future time interval, for example

$$\text{caplet Black's vol 3 months from now} = \left(\frac{1}{T_j - T_1} \int_{T_1}^{T_j} v_j^2\left(t\right)\, dt\right)^{1/2}.$$

Such volatility (called forward volatility) is virtually unobservable yet is very important in the price of some instruments (can you think of any example?)

A trader may wish to control such volatility directly by passing the model a number to match. The formula for forward volatility is straightforward:

$$\text{forward vol between } T \text{ and } S = \left(\frac{1}{S - T} \int_T^S v_j^2\left(t\right)\, dt\right)^{1/2}.$$

6.7. **Forward volatility of a swap rate.** Same idea, but the formula is a bit more complicated.

7. Conclusion

The discovery of market models was a major advance in the modelling of interest rates. The market models allow using market-observed and intuitive quantities in specifying the dynamics of the term structure of interest rates. Whereas the name "market model" suggests a specific set of dynamical equations for interest rates, it is in fact more of a framework than the actual model, whose only difference from any other HJM model is in choosing a different parametrization of the term curve (market rates instead of instantaneous forward ones).

Merely rewriting the equations for market (LIBOR or swap) rates does not constitute a model. To have a model, a set of exogenous constraints should be specified. The market models are convenient to calibrate since prices of caplets, swaptions and other related quantities can readily be expressed in terms of the model parameters (time dependent instantaneous caplet volatilities). Path-dependent instruments can be valued via Monte-Carlo. American-style derivatives (most notably Bermudans) cannot easily be valued because of the lack of low-dimensional Markov parametrization.

8. Appendix

Let us present a simple yet illuminating example on calibrating a market model. It is lifted, pretty much unchanged, from [R2].

Consider a tenor structure

$$\begin{aligned} 0 &= T_0 < T_1 < T_2 < T_3, \\ \delta_m &= T_{m+1} - T_m. \end{aligned}$$

A LIBOR rate L_1 resets at date T_1 and pays at T_2. A LIBOR rate L_2 resets at T_2 and pays at T_3. In addition to these two rates, we have a swap rate S, for a swap that begins at T_1 and pays at T_2 and T_3. In terms of the relevant bonds, we have

$$\begin{aligned} L_1(t) &= \frac{P(t,T_1) - P(t,T_2)}{\delta_1 P(t,T_2)}, \\ L_2(t) &= \frac{P(t,T_2) - P(t,T_3)}{\delta_2 P(t,T_3)}, \\ S(t) &= \frac{P(t,T_1) - P(t,T_3)}{\delta_1 P(t,T_2) + \delta_2 P(t,T_3)}. \end{aligned}$$

We can observe three volatilities from the market. One is the volatility of (the log of) $L_1(t)$ from 0 to T_1. Call it λ_1. The other is the volatility of (the log of) $L_1(t)$ from 0 to T_2, call it λ_2. Finally, from the swaptions's market, we get the volatility of (the log of) $S(t)$ from 0 to T_1, call it ν.

We would like to use this data to "calibrate" a market model for L_1 and L_2. We can assume that in between dates T_m, their instantaneous volatilities are flat. Therefore, to build a model, we have to specify (see Figure 1)

- How the log-normal volatility of L_2 is "split" between $[0, T_1]$ and $[T_1, T_2]$;
- What is the correlation (call it ρ) of L_1 and L_2 over the interval $[0, T_1]$.

As we have shown, we can approximate

$$S(T_1) \approx w_1 L_1(T_1) + w_2 L_2(T_1),$$

where

$$\begin{aligned} w_1 &= \frac{\delta_1 P(0, T_2)}{\delta_1 P(0, T_2) + \delta_2 P(0, T_3)}, \\ w_2 &= \frac{\delta_2 P(0, T_3)}{\delta_1 P(0, T_2) + \delta_2 P(0, T_3)}. \end{aligned}$$

Taking variance of both sides we get (again as before),

$$\operatorname{Var} S(T_1) = w_1^2 \operatorname{Var} L_1(T_1) + w_2^2 \operatorname{Var} L_2(T_1) + 2 w_1 w_2 \operatorname{covar} L_1(T_1) L_2(T_1).$$

Then using the approximation (6.2) we get approximately

$$S^2 \nu^2 T_1 = w_1^2 L_{11}^{2\lambda 2} T_1 + w_2^2 L_2^2 (\lambda_2')^2 T_1 + 2 w_1 w_2 L_1 L_2 \lambda_1 \lambda_2' \rho T_1.$$

Here, everything can be market implied except for ρ (correlation between LIBOR rates over $[0, T_1]$) and log-normal volatility σ_2' of L_2 over $[0, T_1]$. This is a very important point to understand: *this volatility is not available from the market*!

Another equation connects volatility of L_2 over $[0, T_1]$ and $[T_1, T_2]$ (call the latter λ_2''). We have thus two equations,

$$\begin{aligned} S^2 \nu^2 &= w_1^2 L_1^2 \lambda_1^2 + w_2^2 L_2^2 (\lambda_2')^2 + 2 w_1 w_2 L_1 L_2 \lambda_1 \lambda_2' \rho, \\ \lambda_2^2 T_2 &= (\lambda_2')^2 T_1 + (\lambda_2'')^2 (T_2 - T_1). \end{aligned}$$

Two equations and three unknowns (ρ, λ_2', λ_2''). Thus, we have an infinite number of combinations to satisfy the equations,

- Do we take $\rho = 1$ and choose $\lambda_2'' \neq \lambda_2'$; or
- Do we take $\lambda_2'' = \lambda_2'$ and ρ strictly less than 1; or
- Do we make the model time-homogeneous and choose $\lambda_2'' = \lambda_1$; or
- Something in between?

There is virtually no other market information we can use, so it is pretty much a judgement call.

Why the choice is important, however? Consider a periodic (rachet) caplet, an instrument that pays

$$\max\{L_2(T_2) - L_1(T_1), 0\}$$

at time T_3. Market in such instruments is very thin, but clients do ask their brokers to quote prices. It is quite clear that a periodic caplet will derive most of its value from volatility λ_2'', so making a right choice (between perfect correlation and constant volatility) is very important.

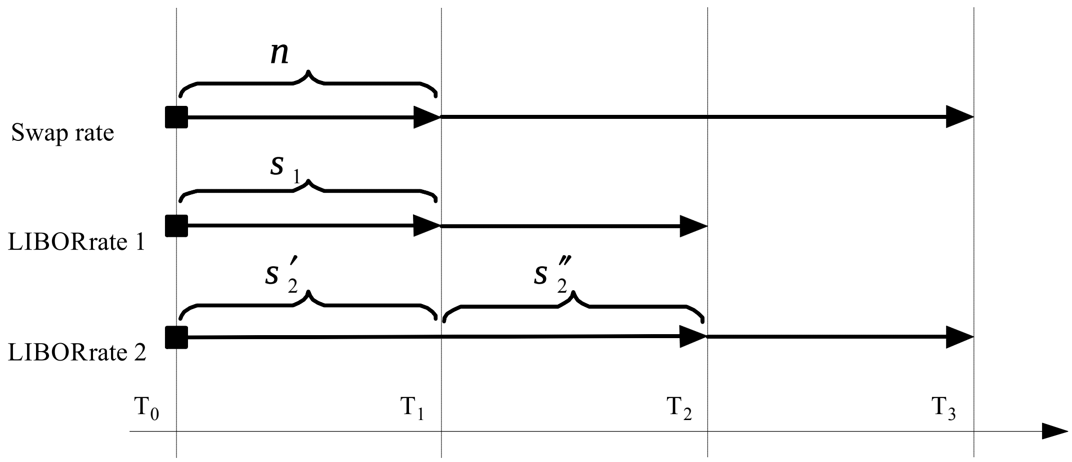

FIGURE 1

Hopefully, this toy example demonstrates the mind-boggling complexity of calibrating a market model.

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